

Black Scholes' Greeks up to 3rd Order

Volume I: Analytical Formulae, Derivation and Implementation using Python and C++

First Edition

AJM, quantpie
quantpie Publishing
books@quantpie.co.uk

Copyright © 2022 quantpie Publishing

All rights reserved.

ISBN: 9798808909380

Contents

Introduction	1
1. First order Derivatives	5
I) Derivative of Price w.r.t S	6
II) Derivative of Price w.r.t K	8
III) Derivative of Price w.r.t Vol	9
IV) Derivative of Price w.r.t t	11
V) Derivative of Price w.r.t r	13
VI) Derivative of Price w.r.t q	14
VII) Summary and Python code	16
2. Derivatives of Delta	19
I) Derivative of Delta w.r.t S	20
II) Derivative of Delta w.r.t K	21
III) Derivative of Delta w.r.t Vol.....	22
IV) Derivative of Delta w.r.t t	23
V) Derivative of Delta w.r.t r	25
VI) Derivative of Delta w.r.t q	25
VII) Summary and Python code.....	27
3. Derivatives of Dual Delta	31
I) Derivative of Dual Delta w.r.t S	32
II) Derivative of Dual Delta w.r.t K	34
III) Derivative of Dual Delta w.r.t Vol.....	35
IV) Derivative of Dual Delta w.r.t t	35
V) Derivative of Dual Delta w.r.t r	37
VI) Derivative of Dual Delta w.r.t q	38
VII) Summary and Python code.....	40

4. Derivatives of Vega	44
I) Derivative of Vega w.r.t S.....	46
II) Derivative of Vega w.r.t K	47
III) Derivative of Vega w.r.t Vol.....	48
IV) Derivative of Vega w.r.t t.....	49
V) Derivative of Vega w.r.t r.....	51
VI) Derivative of Vega w.r.t q.....	52
VII) Summary and Python code.....	54
5. Derivatives of Theta	58
I) Derivative of Theta w.r.t S	60
II) Derivative of Theta w.r.t K	63
III) Derivative of Theta w.r.t Vol.....	65
IV) Derivative of Theta w.r.t t.....	67
V) Derivative of Theta w.r.t r	70
VI) Derivative of Theta w.r.t q	72
VII) Summary and Python code.....	75
6. Derivatives of Rho	80
I) Derivative of Rho w.r.t S	81
II) Derivative of Rho w.r.t K	83
III) Derivative of Rho w.r.t Vol	84
IV) Derivative of Rho w.r.t t.....	85
V) Derivative of Rho w.r.t r	86
VI) Derivative of Rho w.r.t q	87
VII) Summary and Python code.....	89
7. Derivatives of Epsilon	93
I) Derivative of Epsilon w.r.t S	94
II) Derivative of Epsilon w.r.t K.....	95
III) Derivative of Epsilon w.r.t Vol	96
IV) Derivative of Epsilon w.r.t t	97
V) Derivative of Epsilon w.r.t r.....	99
VI) Derivative of Epsilon w.r.t q	100
VII) Summary and Python code.....	102

8. Derivatives of Gamma	106
I) Derivative of Gamma w.r.t S	107
II) Derivative of Gamma w.r.t K.....	108
III) Derivative of Gamma w.r.t Vol	109
IV) Derivative of Gamma w.r.t t	110
V) Derivative of Gamma w.r.t r	112
VI) Derivative of Gamma w.r.t q	113
VII) Summary and Python code	115
9. Derivatives of Dual Gamma	119
I) Derivative of Dual Gamma w.r.t S	120
II) Derivative of Dual Gamma w.r.t K.....	122
III) Derivative of Dual Gamma w.r.t Vol	123
IV) Derivative of Dual Gamma w.r.t t	124
V) Derivative of Dual Gamma w.r.t r.....	126
VI) Derivative of Dual Gamma w.r.t q	127
VII) Summary and Python code.....	129
10. Derivatives of Volga	133
I) Derivative of Volga w.r.t S	135
II) Derivative of Volga w.r.t K.....	137
III) Derivative of Volga w.r.t Vol	138
IV) Derivative of Volga w.r.t t.....	140
V) Derivative of Volga w.r.t r.....	142
VI) Derivative of Volga w.r.t q	143
VII) Summary and Python code.....	146
11. Derivatives of Time-Gamma	150
I) Derivative of Time Gamma w.r.t S	151
II) Derivative of Time Gamma w.r.t K.....	154
III) Derivative of Time Gamma w.r.t Vol	157
IV) Derivative of Time Gamma w.r.t t.....	161
V) Derivative of Time Gamma w.r.t r	167
VI) Derivative of Time Gamma w.r.t q	170
VII) Summary and Python code	173

12. Derivatives of Discount Rate Gamma	180
I) Derivative of Discount Gamma w.r.t S	182
II) Derivative of Discount Gamma w.r.t K	183
III) Derivative of Discount Gamma w.r.t Vol.....	185
IV) Derivative of Discount Gamma w.r.t t	187
V) Derivative of Discount Gamma w.r.t r	189
VI) Derivative of Discount Gamma w.r.t q	191
VII) Summary and Python code	193
13. Derivatives of Dividend Gamma.....	197
I) Derivative of Dividend Gamma w.r.t S	199
II) Derivative of Dividend Gamma w.r.t K	200
III) Derivative of Dividend Gamma w.r.t Vol.....	202
IV) Derivative of Dividend Gamma w.r.t t	203
V) Derivative of Dividend Gamma w.r.t r	206
VI) Derivative of Dividend Gamma w.r.t q	207
VII) Summary and Python code.....	209
14. Mixed Distinct	213
I) Derivative of Price w.r.t S , Vol, and K	215
II) Derivative of Price w.r.t S , Vol, and t	216
III) Derivative of Price w.r.t S , Vol, and r	218
IV) Derivative of Price w.r.t S , Vol, and q	219
V) Derivative of Price w.r.t S , K , and t	221
VI) Derivative of Price w.r.t S , K , and r	223
VII) Derivative of Price w.r.t S , K , and q	224
VIII) Derivative of Price w.r.t S , r , and t	226
IX) Derivative of Price w.r.t S , r , and q	228
X) Derivative of Price w.r.t S , q , and t	229
XI) Derivative of Price w.r.t Vol, K , and t	232
XII) Derivative of Price w.r.t Vol, K , and r	234
XIII) Derivative of Price w.r.t Vol, K , and q	236
XIV) Derivative of Price w.r.t Vol, r , and t	237
XV) Derivative of Price w.r.t Vol, r , and q	239
XVI) Derivative of Price w.r.t Vol, q , and t	240

XVII) Derivative of Price w.r.t K , r , and t	242
XVIII) Derivative of Price w.r.t K , r , and q	245
XIX) Derivative of Price w.r.t K , q , and t	246
XX) Derivative of Price w.r.t r , q , and t	248
XXI) Summary and Python code	251
Appendix A: Derivatives of $d1$ and $d2$	261
Appendix B: Standard Normal.....	268
Appendix C: An Identity	272
Appendix D: Differentiation Rules	274
I) Derivative of a constant function	275
II) Derivative of a constant times a function	275
III) Derivative of sum of functions	275
IV) Derivative of linear combination of functions	276
V) Derivative of product of functions	276
VI) Derivative of ratio of two functions	277
VII) Chain rule	278
VIII) Derivative of exponential function	279
IX) Derivative of logarithmic function.....	280
X) Derivative of power function	281
XI) Derivative of $\exp(ax)$	283
Appendix E: Finite Difference	285
Appendix F: C++ Code	293
References	315
Index	316

Preface

The purpose of this volume is to derive analytical formulae for the derivatives, or what have become known as Greeks, of the Black Scholes' formula. Whilst the formulae (and their derivations) for the first order derivatives are widely available, the second and third order derivatives are sparsely covered in the existing literature. This work covers the derivations of the formulae for the first order, second order, and third order derivatives, and thus aims to serve as a complete handbook for the derivatives of the Black Scholes' formula up to the third order. We also implement these formulae using Python and C++, and validate the results using the Finite Difference approach. The second volume will cover the numerical approaches, e.g., Finite Difference, Algorithmic Differentiation, and Monte Carlo methods, for calculating these derivatives.

We assume familiarity with the Black Scholes' formula, and rules for differentiation, though the differentiation rules we use are briefly covered in Appendix E, mainly to settle the terminology. We do not expect that all readers will read the book from front to back, page by page, and we have therefore made attempts to make the chapters independent of each other. This leads to some repetitions, which we have compensated for, to some extent, by delegating the derivation steps we encounter in almost all of the derivations to appendices. The appendices helped reduce the size of the book, but luckily did not add too many pages, which is a relief because we expect that these constitute essential read!

In each derivation, we have attempted to include as many steps as needed to ensure that readers can replicate the derivations. We sometimes use more than one operation in a step, but we have tried to group the operations in such a way that the application of each rule can be discerned from visual inspection of any two consecutive lines. Of course, this is a subjective statement, and at times it may appear that we have applied many basic rules in the same step, or should have combined two steps. The commentary provides reference to the relevant rules, though again we do appreciate that the length of commentary is a subjective matter, and thus some readers would have preferred more commentary. If these aspects are of interest to you, then please do let us know so that we can improve the derivation steps and the quality of the commentary in the future editions.

All derivations follow similar approach in that we use the basic differentiation rules in all of them - that is, start with a formula, differentiate it with respect to the variables of interest, and simplify it using differentiation and algebraic rules. We are aware of some tricks that simplifies the derivation of some of the derivatives; however, we don't pursue these approaches in this volume. There are also numerous other approaches such as series type representation, derivatives defined in terms of some basic Greeks, or as solution of some PDEs. However, we don't cover any of these topics, we patiently employ the same differentiation based approach in all of the derivations.

Once the formulae are derived, we implement them in Python and C++. The Python code is in the last section of each chapter, and each formula is covered in a standalone function. The C++ code is in an appendix, and we have grouped the functions inside a class. This helps us reduce the calculations as the common calculations are performed once, and are not repeated in every function. We could have grouped the Python functions as well, but we kept them separate for readers who might be interested in a specific formula. To test the accuracy of the formulae, we compared the results produced by the Python and the C++ code with the results we generated using Finite Difference (FD) approach. For the comparisons performed in this volume, we have used the second order accurate formulae which we derive in Volume II. Volume II also covers validation against other numerical approaches.

The usual disclaimers follow. The code is included for illustration, and is not meant to be production code. Additionally, the code includes very few comments as we expect the readers to be familiar with either Python or C++. We have tried not to use too many obscure functionalities or idioms, and have not fully optimised the code. But hopefully it is relatively easy to follow, which has been our guiding principle.

Your feedback will make it better! Please email books@quantpie.co.uk for comments, feedback, or any questions about this book.